

Compact spacelike surfaces whose mean curvature function satisfies a nonlinear inequality in a 3-dimensional Generalized Robertson-Walker spacetime

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Abstract

Spacelike surfaces in Generalized Robertson-Walker spacetimes whose mean curvature function satisfies a natural nonlinear inequality are analyzed. Several uniqueness and nonexistence results for such compact spacelike surfaces are proved. In the nonparametric case, new Calabi-Bernstein type problems are solved as a consequence.

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1 Introduction

Let $f : I \rightarrow \mathbb{R}$ be a positive smooth function on an open interval $I =]a, b[$, $-\infty \leq a < b \leq \infty$, of the real line \mathbb{R} and let (F, g) be a 2-dimensional Riemannian manifold. For each $u \in C^\infty(\Omega)$, Ω an open domain in F , such that $u(\Omega) \subset I$ and $|Du| < f(u)$, where $|Du|$ stands for the length of the gradient Du of u , we consider the smooth function

$$H(u) = -\operatorname{div} \left(\frac{Du}{2f(u)\sqrt{f(u)^2 - |Du|^2}} \right) - \frac{f'(u)}{2\sqrt{f(u)^2 - |Du|^2}} \left(2 + \frac{|Du|^2}{f(u)^2} \right), \quad (1)$$

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where div represents the divergence operator on (F, g) . When $f = 1$ (constant), and $u \in C^\infty(\Omega)$, Ω an open domain in F , such that $|Du| < 1$, then $H(u)$ takes the more familiar form

$$H(u) = -\text{div} \left(\frac{Du}{\sqrt{1 - |Du|^2}} \right),$$

and, now, $H(u)$ is the mean curvature function of the graph $\Sigma_u = \{(u(p), p) : p \in \Omega\}$ of u , which is spacelike because $|Du| < 1$, in the Lorentzian product $I \times F$ with respect to the unit normal vector field

$$N = -\frac{1}{\sqrt{1 - |Du|^2}} (1, Du).$$

In order to give an interpretation of $H(u)$ in the general case, consider $M = I \times F$ endowed with the Lorentzian metric

$$\langle \cdot, \cdot \rangle = -\pi_I^*(dt^2) + f(\pi_I)^2 \pi_F^*(g), \quad (2)$$

where π_I and π_F denote the projections onto the open interval I of \mathbb{R} and F , respectively; g is the Riemannian metric of F and $f > 0$ is a smooth function on I . The Lorentzian manifold $(M, \langle \cdot, \cdot \rangle)$ is a warped product (in the terminology of [9, p. 204]) with base $(I, -dt^2)$, fiber (F, g) and warping function f . We will refer to $(M, \langle \cdot, \cdot \rangle)$ as a (3-dimensional) *Generalized Robertson-Walker* (GRW) spacetime (extending the classical notion of Robertson-Walker spacetime to the case when (F, g) is not assumed to have constant Gauss curvature), [2].

For each $u \in C^\infty(\Omega)$, $u(\Omega) \subset I$, the induced metric on Ω , via the graph $\{(u(p), p) : p \in \Omega\} \subset M$, is written on Ω as follows,

$$g_u = -du^2 + f(u)^2 g, \quad (3)$$

which is positive definite, if and only if u satisfies $|Du| < f(u)$. In this case,

$$N = -\frac{1}{f(u)\sqrt{f(u)^2 - |Du|^2}} (f(u)^2, Du), \quad (4)$$

is a unit normal vector field and the function $H(u)$, is the mean curvature, with respect to N , for the spacelike graph of u in M .

Our objective here is to state several uniqueness and nonexistence results for entire solutions to the following nonlinear elliptic problem:

$$H(u)^2 \leq \frac{f'(u)^2}{f(u)^2}, \quad (I.1)$$

$$|Du| < f(u), \quad (I.2)$$

where, at each value u_0 of u , the right hand side of (I.1) is, the squared mean curvature of the spacelike slice $t = u_0$.

This problem, has been studied by the authors in the particular case in which F is the Euclidean plane \mathbb{R}^2 [13]. In the present work, we will deal with (I) when F is a compact Riemannian surface and use a different approach to the one in [13]. In fact, in that paper a local integral estimation of the length of the gradient of the restriction of the warping function

on a (necessarily noncompact) spacelike surface is proved in the parametric case. Then, this is used to obtain uniqueness results both in the parametric and nonparametric case (see Remark 3.3). Here, we will previously study compact spacelike surfaces S in a GRW spacetime M such that its mean curvature function H satisfies the inequality

$$H^2 \leq \frac{f'(t)^2}{f(t)^2}, \quad (\tilde{\text{I}})$$

on all S . Let us remark that if a GRW spacetime admits a compact spacelike surface, then its fiber is necessarily compact, in this case, the spacetime is called *spatially closed* (Remark 2.1). Our approach here is global, in fact under suitable assumptions on M , the inequality $(\tilde{\text{I}})$ implies, making use of (11), the existence of a superharmonic function on S whose constancy permits to classify the compact spacelike surfaces which satisfy $(\tilde{\text{I}})$ in several distinguished cases. It should be pointed out that although formula (11) may be extended to higher dimensions, its meaning changes drastically for $n \geq 3$ (Remark 2.3).

Later, we will derive several results in the nonparametric case, i.e., for entire solutions u to (I) on certain compact Riemannian surfaces.

Several comments on (I) (and $(\tilde{\text{I}})$) are now in order: **(1)** This inequality means that at any value u_0 , of u , $|H(u_0)|$ does not exceed the analogous quantity for the spacelike slice $t = u_0$. **(2)** But it is not a comparison assumption between extrinsic quantities of two spacelike surfaces of M (the right member is, at each u_0 , the squared mean curvature of $t = u_0$). **(3)** It has a real sense if f is not constant (if f is constant, then it is clearly equivalent to $H = 0$). **(4)** In any GRW spacetime, $(\tilde{\text{I}})$ is automatically satisfied by any of its maximal surfaces, and, under some extra assumption, by any of its complete constant mean curvature spacelike surfaces [12]. **(5)** The inequality may be physically interpreted as follows: when F is compact, equation (I.2) reads $|Du| < \lambda f(u)$, where λ is constant with $0 < \lambda < 1$. Therefore, we get $\cosh \theta < \frac{1}{\sqrt{1-\lambda^2}}$, where θ is the hyperbolic angle between N and $-\partial_t$. Along S there exist two families of instantaneous observers \mathcal{T}_p , $p \in S$, where $\mathcal{T}_p = -\partial_t(p)$ (the sign minus depends of the time orientation chosen here) and the normal observers N_p , $p \in S$. The quantities $\cosh \theta(p)$ and $v(p) := (\frac{1}{\cosh \theta(p)})N_p^F$ are respectively the *energy* and the *relative velocity* that \mathcal{T}_p measures for N_p , and the speed satisfies $|v| = \tanh \theta$ on S , [14, 2.1.3]. Therefore, we have $|v| < \lambda$ and hence $|v|$ does not approach to speed of light in vacuum ($= 1$) on S . On the other hand, if the GRW spacetime is a perfect fluid, the restriction of the total energy on any compact spacelike surface, under the inequality assumption, is bounded from above in terms of topological and extrinsic quantities (see Remark 2.4).

Observe that the mean curvature of the spacelike slice $t = t_0$ is the constant $-\frac{f'(t_0)}{f(t_0)}$, so in the parametric case, is natural to wonder,

Under which assumptions, the spacelike slices $t = t_0$ are characterized by the inequality $(\tilde{\text{I}})$?

On the other hand, For each $t_0 \in I$, the spacelike graph defined by the constant function $u = t_0$ satisfies inequality (I), with $H = -\frac{f'(t_0)}{f(t_0)}$, thus, it is also natural to wonder,

Under which assumptions, the constant functions $u = t_0$ are the only entire solutions of the inequality (I)?

Our main aim in this paper is to state several answers, under suitable assumptions on the warping function f with a clear geometric meaning. In order to do that, we will work directly on (immersed) spacelike surfaces in a 3-dimensional GRW spacetime M instead of space-like graphs. Recall that a spacelike surface in M is locally a spacelike graph and this holds globally under some extra topological assumptions [2, Section 3]. Let remark that the three dimensional GRW spacetimes may be seen as a suitable family of toy cosmological models where to test mathematical properties with a potential extension to realistic four dimensional spacetimes. The assumption “spatially closed” has a well-know physical meaning: the spatial universe seems to be finite for a distinguished family of observers, which is physically reasonable at least as a first attempt to model spacetime. In a spatially closed GRW spacetime, the spacelike slices are (finite) observed spatial universes. To state when spacelike slices are the only compact spacelike surfaces which satisfies the nonlinear inequality is just our main goal. In fact, we prove (Theorem 3.1),

Let S be a compact spacelike surface of a GRW spacetime, whose warping function satisfies $(\log f)'' \leq 0$. If the mean curvature H of S satisfies (\tilde{I}) , then S is a spacelike slice.

The convexity assumption of the function $-\log f$ in this result (and in those that follow) is automatically fulfilled if the curvature of the GRW spacetime satisfies a natural condition with a clear physical meaning (see Subsection 2.1).

Observe that no prescribed behavior for the Gauss curvature K^F of F is assumed in previous results. Under a boundedness assumption of K^F , we prove (Corollary 3.5),

Let M be a GRW spacetime, whose warping function satisfies $(\log f)'' \leq 0$ and the Gauss curvature of its fiber obeys $K^F(\pi_F) \geq cf(\pi_F)^2$ for some constant $c > 0$. If M admits a complete spacelike surface S such that the inequality (\tilde{I}) , holds on all S , then M must be spatially closed and S is a spacelike slice.

As a consequence, we get (Corollary 3.6),

Let M be a GRW spacetime whose warping function satisfies $(\log f)'' \leq 0$. Let S a compact spacelike surface in M and put $\alpha = \min_S \frac{f'(t)^2}{f(t)^2}$. If the mean curvature function H of S satisfies

$$H^2 \leq \alpha,$$

then S must be a spacelike slice. Moreover, there is no compact spacelike surface in M whose mean curvature satisfies $H^2 < \alpha$.

Moreover, we also give a nonexistence result (Theorem 3.7) and a uniqueness result for the case of a RW spacetime of non-positive constant sectional curvature (Theorem 3.10).

As consequence of these parametric theorems we obtain several new uniqueness as well as a nonexistence results of Calabi-Bernstein type (Theorems 4.1 and 4.2)

Let F be a 2-dimensional compact Riemannian manifold and let $f : I \rightarrow]0, \infty[$ be a smooth function such that $(\log f)'' \leq 0$. Then, the only entire solutions $u : F \rightarrow I$ of the inequality (I) are the constant functions. Moreover, there is no entire solution of the inequality assuming that (I.1) holds strictly.

Finally, we obtain the following uniqueness and nonexistence result,

Let F be a 2-dimensional compact Riemannian manifold and let $f : I \rightarrow]0, \infty[$ be a smooth function with $(\log f)'' \leq 0$ and such that $\inf \frac{(f')^2}{f^2} = \alpha > 0$. Then, the only entire solutions $u : F \rightarrow I$ of the inequality

$$H(u)^2 \leq \alpha,$$

$$|Du| < f(u),$$

are the constant functions. Moreover, there is no entire solution if the first inequality is strict.

2 Preliminaries

Let f be a positive smooth function defined on an open interval I of \mathbb{R} and let (F, g) be a Riemannian surface. Consider $M = I \times F$ endowed with the Lorentzian metric $\langle \cdot, \cdot \rangle$ given in (2), i.e., M is a GRW spacetime as defined in previous section. The vector field $\partial_t := \partial/\partial t \in \mathfrak{X}(M)$ is timelike (and unitary) and hence M is time orientable. Along this paper, when a time orientation is necessary to choose on M , we agree to consider M endowed with the time orientation defined by $-\partial_t$.

The spacetime M has a distinguished infinitesimal conformal symmetry, namely the vector field $\xi := f(\pi_I) \partial_t$, is timelike and, from the relationship between the Levi-Civita connections of M and those of the base and the fiber [9, Cor. 7.35], satisfies

$$\overline{\nabla}_X \xi = f'(\pi_I) X, \tag{5}$$

for any $X \in \mathfrak{X}(M)$, where $\overline{\nabla}$ is the Levi-Civita connection of the metric (2). Thus, ξ is conformal with $\mathcal{L}_\xi \langle \cdot, \cdot \rangle = 2f'(\pi_I) \langle \cdot, \cdot \rangle$ and its metrically equivalent 1-form is closed.

2.1 Energy conditions

Recall that a Lorentzian manifold obeys the *timelike convergence condition* (TCC) if its Ricci tensor $\overline{\text{Ric}}$ satisfies

$$\overline{\text{Ric}}(Z, Z) \geq 0,$$

for all timelike tangent vector Z . It is normally argued that TCC on a spacetime is the mathematical way to express that gravity, on average, attracts [9, p. 340]. A weaker energy condition is the *null convergence condition* (NCC) which reads

$$\overline{\text{Ric}}(Z, Z) \geq 0,$$

for any *null* tangent vector Z , i.e. $Z \neq 0$ satisfying $\langle Z, Z \rangle = 0$. Clearly, a continuity argument shows that TCC implies NCC. A spacetime M obeys the *ubiquitous energy condition* if

$$\overline{\text{Ric}}(Z, Z) > 0,$$

for all timelike tangent vector Z . This last energy condition is clearly stronger than TCC and roughly means a real presence of matter at any event of the spacetime.

It is easy to see that a GRW spacetime M with a 2-dimensional fiber (F, g) obeys TCC if and only if its warping function satisfies $f'' \leq 0$ and the Ricci tensor of the fiber satisfies $\text{Ric}^F \geq (ff'' - f'^2)g$, [2]. Moreover, TCC holds if and only if NCC hold and $f'' \leq 0$. On the other hand, if a GRW spacetime M obeys the ubiquitous energy condition, then $f'' < 0$. Observe that if $f'' \leq 0$ (resp. $f'' < 0$) then $(\log f)'' \leq 0$ (resp. $(\log f)'' < 0$).

From [9, Cor. 7.43], we have

$$\overline{\text{Ric}}(X, Y) = \text{Ric}^F(X^F, Y^F) + \left(\frac{f''}{f} + \frac{(f')^2}{f^2} \right) \langle X^F, Y^F \rangle - \frac{2f''}{f} \langle X, \partial_t \rangle \langle Y, \partial_t \rangle, \quad (6)$$

for any tangent vectors X, Y to M , where $Z^F := Z + \langle Z, \partial_t \rangle \partial_t$ is the projection of the tangent vector Z on the fiber. Therefore,

$$\overline{\text{Ric}}(Z, Z) = \left\{ \frac{K^F(\pi_F)}{f^2} - (\log f)'' \right\} \langle Z, \partial_t \rangle^2,$$

in the case Z is null, where K^F denotes the Gauss curvature of F . Thus, the GRW spacetime M obeys NCC if and only if

$$\frac{K^F(\pi_F)}{f^2} - (\log f)'' \geq 0. \quad (7)$$

Along the main results of this paper we will assume the warping function of the GRW spacetime M satisfies $(\log f)'' \leq 0$. Note that, in particular, this holds whenever M obeys TCC. Moreover, if $K^F \geq 0$, the inequality $(\log f)'' \leq 0$ implies NCC. On the other hand, if $K^F \leq 0$ holds, then NCC implies $(\log f)'' \leq 0$. In addition, when the spacetime obeys the ubiquitous energy condition the inequality $(\log f)'' < 0$ is satisfied.

2.2 The restriction of the warping function on a spacelike surface

Let $x : S \rightarrow M$ be a (connected) spacelike surface in M , i.e., x is an immersion and it induces a Riemannian metric on the 2-dimensional manifold S from the Lorentzian metric (2). As usual, we agree to represent the induced metric with the same symbol as the one used in (2). Then the time-orientability of M allows us to consider $N \in \mathfrak{X}^\perp(S)$ as the only, globally defined, unitary timelike normal vector field on S in the same time-orientation of $-\partial_t$. Thus, from the wrong way Cauchy-Schwarz inequality, (see [9, Prop. 5.30], for instance) we have $\langle N, \partial_t \rangle \geq 1$ and $\langle N, \partial_t \rangle = 1$ at a point p if and only if $N(p) = -\partial_t(p)$.

In any GRW spacetime M , the level surfaces of the function $\pi_I : M \rightarrow I$ constitute a distinguished family of spacelike surfaces, the so-called *spacelike slices*. We agree to represent by $t = t_0$ the spacelike slice $\{t_0\} \times F$. For a given spacelike surface $x : S \rightarrow M$, we have that $x(S)$ is contained in $t = t_0$ if and only if $\pi_I \circ x = t_0$ on S . Note that this holds if and only if the surface S is orthogonal to ∂_t or, equivalently, orthogonal to ξ . We will say that S is a spacelike slice if $x(S)$ equals to $t = t_0$, for some $t_0 \in I$.

Denote by $\partial_t^\top := \partial_t + \langle N, \partial_t \rangle N$ the tangential component of ∂_t on S . From (5) using the Gauss formula, we have

$$\nabla t = -\partial_t^\top, \quad (8)$$

where ∇t is the gradient of $t := \pi_I \circ x$. Now, using again the Gauss formula, taking into account $\xi^\top = f(t) \partial_t^\top$ and (8), the Laplacian of the function t on S satisfies

$$\Delta t = -\frac{f'(t)}{f(t)} \left\{ 2 + |\nabla t|^2 \right\} - 2H \langle N, \partial_t \rangle, \quad (9)$$

where $f(t) := f \circ t$, $f'(t) := f' \circ t$ and $H := -(1/2) \text{trace}(A)$ is called the *mean curvature* function of S relative to N , where A is the shape operator associated to N . A spacelike surface S with constant mean curvature is a critical point of the area functional under a certain volume constraint (see [5], for instance). A spacelike surface with constant H is called a spacelike surface of constant mean curvature surface (CMC). Note that, with our choice of N , the shape operator of the spacelike slice $t = t_0$ is $A = (f'(t_0)/f(t_0)) I$ and $H = -f'(t_0)/f(t_0)$, therefore the spacelike slices are totally umbilical CMC spacelike surfaces.

A direct computation from (8) and (9) gives

$$\Delta f(t) = -2 \frac{f'(t)^2}{f(t)} + f(t)(\log f)''(t) |\nabla t|^2 - 2f'(t)H \langle N, \partial_t \rangle, \quad (10)$$

for any spacelike surface S in the GRW spacetime M .

Remark 2.1 For any spacelike surface S the mapping $\pi_F \circ x$ is a local diffeomorphism, which is a covering map when S is complete and $f(t)$ is bounded on S [2, Lemma 3.1]. Therefore if S is compact, then F must be also compact.

Remark 2.2 As explained in the Introduction, the graph in M of $u \in C^\infty(F)$, which satisfies $u(F) \subset I$ and $|Du| < f(u)$ is a spacelike surface in M . Each spacelike surface is locally the graph of a function, but it is not true globally in general. However, if F is assumed to be compact and simply connected, then every compact spacelike surface in M must be diffeomorphic to F and $x(S)$ may be seen as a spacelike graph in M , [2, Prop. 3.3].

From (10) and taking into account $-1 = |\nabla t|^2 - \langle N, \partial_t \rangle^2$, the Laplacian of the function $\log f(t)$ on S satisfies

$$\Delta \log f(t) = - \left(\frac{f'(t)}{f(t)} + H \langle N, \partial_t \rangle \right)^2 + \left(H^2 - \frac{f'(t)^2}{f(t)^2} \right) \langle N, \partial_t \rangle^2 + (\log f)''(t) |\nabla t|^2. \quad (11)$$

Observe that, under the assumption $(\log f)''(t) \leq 0$, the inequality (\tilde{I}) gives that $\log f(t)$ is superharmonic on S . In particular, this holds on each spacelike surface in M whose mean curvature function satisfies the inequality (\tilde{I}) , whenever M obeys TCC or NCC with $K^F \leq 0$.

Remark 2.3 In view of previous assertion, equation (11) is a key fact for the obtainment of the results in this paper. It is then natural to wonder if (11) may be stated for a spacelike hypersurface in an $(n+1)$ -dimensional GRW spacetime, for any $n \geq 2$. In this general case, it is not difficult to get,

$$\begin{aligned} \Delta \log f(t) = & - \left(\frac{f'(t)}{f(t)} + H \langle N, \partial_t \rangle \right)^2 + \left(H^2 - \frac{f'(t)^2}{f(t)^2} \right) \langle N, \partial_t \rangle^2 + (\log f)''(t) |\nabla t|^2 \\ & - (n-2) \frac{f'(t)^2}{f(t)^2} - (n-2) \frac{f'(t)}{f(t)} H \langle N, \partial_t \rangle. \end{aligned}$$

2.3 The Gauss curvature of a spacelike surface

Denote by R and \bar{R} the curvature tensors of a spacelike surface S and of M , respectively. The Gauss equation reads

$$\langle R(X, Y)U, V \rangle = \langle \bar{R}(X, Y)U, V \rangle - \langle AY, U \rangle \langle AX, V \rangle + \langle AX, U \rangle \langle AY, V \rangle, \quad (12)$$

where $X, Y, U, V \in \mathfrak{X}(S)$. Moreover, we have the Codazzi equation which, taking into account that the normal bundle of the spacelike surface is negative definite, is written as follows

$$\bar{R}(X, Y)N = -(\nabla_X A)Y + (\nabla_Y A)X, \quad (13)$$

for all $X, Y \in \mathfrak{X}(S)$.

From the Gauss equation (12) we get

$$\text{Ric}(X, Y) = \bar{\text{Ric}}(X, Y) + \langle \bar{R}(N, X)Y, N \rangle + 2H \langle AX, Y \rangle + \langle A^2 X, Y \rangle, \quad (14)$$

where Ric denotes the Ricci tensor of S .

Now we take a local orthonormal frame field E_1, E_2, E_3 on M which is adapted to S , i.e., on S , E_1, E_2 are tangent to S and $E_3 = N$. From (14) we obtain

$$2K = \sum_{i=1}^2 \bar{\text{Ric}}(E_i, E_i) + \sum_{i=1}^2 \langle \bar{R}(N, E_i)E_i, N \rangle - 4H^2 + \text{trace}(A^2), \quad (15)$$

where K is the Gauss curvature of S . We can rewrite (15), using [9, Prob. 7.13], as follows

$$K = \frac{f'(t)^2}{f(t)^2} + \left\{ \frac{K^F(\pi_F)}{f(t)^2} - (\log f)''(t) \right\} |\nabla t|^2 + \frac{K^F(\pi_F)}{f(t)^2} - 2H^2 + \frac{1}{2} \text{trace}(A^2). \quad (16)$$

The Schwarz inequality for symmetric operators on a 2-dimensional euclidean vector space implies $H^2 \leq \frac{1}{2} \text{trace}(A^2)$. Taking this in mind, if we assume M obeys NCC and the inequality $(\tilde{\text{I}})$ holds on S , then (16) implies $K \geq \frac{K^F(\pi_F)}{f(t)^2}$, i.e., at each $p \in S$, $K(p)$ is at least the Gauss curvature of the slice $t = t(p)$ at the point $\pi_F(x(p))$.

Remark 2.4 Assume the GRW spacetime M is a (3-dimensional) *perfect fluid* with flow vector field $-\partial_t$ and *energy density function* ρ [9, Def. 12.4]. From (6) we have

$$8\pi\rho = \frac{K^F}{f^2} + \frac{(f')^2}{f^2}. \quad (17)$$

Now consider a spacelike surface S in M . Using that M satisfies NCC, from (16) and previous formula we have

$$K \geq 8\pi\rho - H^2. \quad (18)$$

on each spacelike surface S in M . If S is in addition assumed to be compact, the *total energy* on S is

$$E_S := \int_S \rho dS.$$

The inequality $(\tilde{\text{I}})$ permits to deduce from (18), via the Gauss-Bonnet theorem the following upper bound for the total energy on S ,

$$E_S \leq \frac{1}{2} \mathcal{X}(S) + \frac{1}{8\pi} \int_S \frac{f'(t)^2}{f(t)^2} dS, \quad (19)$$

where $\mathcal{X}(S)$ denotes the Euler number of S .

3 Uniqueness results in the parametric case

We begin with a direct consequence of formula (11),

Theorem 3.1 *Let S be a compact spacelike surface of a GRW spacetime, whose warping function satisfies $(\log f)'' \leq 0$. If the mean curvature function H of S satisfies (\tilde{I}) , then S is a spacelike slice.*

Proof. The assumptions on f and H clearly give $\Delta \log f(t) \leq 0$ making use of (11). The compactness of S implies that the function $\log f(t)$ is constant on S , and, therefore, $f(t)$ is so. Consider now a primitive function \mathcal{F} of f and write $\mathcal{F}(t)$ for the restriction of $\mathcal{F} \circ \pi_I$ on S . Note that $\nabla \mathcal{F}(t) = f(t) \nabla t$. Observe that the vanishing of the first term on the right of (11) means $\Delta \mathcal{F}(t) = 0$. Consequently, $\mathcal{F}(t)$ is constant, and then, S is a spacelike slice. \square

Remark 3.2 Note that no prescribed behavior for the Gauss curvature K^F of F is assumed in Theorem 3.1. On the other hand, the assumption $(\log f)'' \leq 0$ can not be removed. To support this assertion consider $I = \mathbb{R}$, $f(t) = \cosh(t)$ and $F = \mathbb{S}^2$ with its canonical metric of constant Gauss curvature 1, then the corresponding 3-dimensional GRW spacetime M is the De Sitter spacetime \mathbb{S}_1^3 of constant sectional curvature 1. It is well-known that any compact maximal surface in the 3-dimensional De Sitter spacetime must be totally geodesic [11], and not any compact totally geodesic spacelike surface in \mathbb{S}_1^3 is a spacelike slice.

Remark 3.3 The compactness of S in Theorem 3.1 cannot be relaxed to completeness in general. In fact, a complete maximal surface in Lorentz-Minkowski spacetime \mathbb{L}^3 is a spacelike plane (see for instance [8]), and, clearly, not any spacelike plane is a spacelike slice in \mathbb{L}^3 . However, complete spacelike surfaces in certain GRW spacetimes, which are far from Lorentzian products, have been studied in [13]. In fact, in that reference the warping function is assumed non locally constant, i.e., there is no open rectangle $J \times F$ in M such that, the restriction of the Lorentzian metric (2) to $J \times F$ is a product. Such a GRW spacetime is called *proper*.

Remark 3.4 Taking into account [1, Th. 3.1], [7, Cor. 5.10], the mean curvature H_0 of a compact CMC spacelike surface in a GRW spacetime, whose warping function satisfies $(\log f)'' \leq 0$, satisfies (\tilde{I}) (really, the equality holds). Alternatively, this fact can be deduced from the argument which follows. In fact, denote as previously by \mathcal{F} a primitive function of f and by $\mathcal{F}(t)$ its restriction on S . From (9) we get

$$\Delta \mathcal{F}(t) = -2f'(t) - 2H_0 f(t) \langle N, \partial_t \rangle.$$

Let p_0 and p^0 be the points of S where $\mathcal{F}(t)$ attains its global minimum and maximum values, respectively. We have $\nabla t(p_0) = \nabla t(p^0) = 0$ and $\langle N, \partial_t \rangle(p_0) = \langle N, \partial_t \rangle(p^0) = 1$. On the other hand, $\Delta \mathcal{F}(t)(p_0) \geq 0$ and $\Delta \mathcal{F}(t)(p^0) \leq 0$ hold, and therefore the constant H_0 satisfies

$$\frac{-f'(t(p^0))}{f(t(p^0))} \leq H_0 \leq \frac{-f'(t(p_0))}{f(t(p_0))}.$$

Now, it is enough to observe that the function \mathcal{F} is increasing from its definition, and the function $-f'/f$ is also increasing from our assumption, giving the equality $H_0 = -f'(t)/f(t)$. Therefore, for the case of 3-dimensional GRW spacetimes, the previously quoted result in [1] follows from Theorem 3.1.

The same argument does not work for the case of noncompact complete spacelike surfaces. In fact, as an application of the classical generalized maximum principle, due to Omori [10] and Yau [15], it was shown, [12, Prop. 5.3], that a complete spacelike surface of constant mean curvature in certain proper RW spacetimes with fiber \mathbb{R}^2 , warping function satisfying $(\log f)'' \leq 0$ and which is contained between two spacelike slices must satisfy (\tilde{I}) .

Corollary 3.5 *Let M be a GRW spacetime, whose warping function satisfies $(\log f)'' \leq 0$ and the Gauss curvature of its fiber obeys $K^F(\pi_F) \geq c f(\pi_F)^2$ for some constant $c > 0$. If M admits a complete spacelike surface S such that the inequality (\tilde{I}) holds on all S , then M is spatially closed and S is a spacelike slice.*

Proof. From (16), the Gauss curvature of S satisfies $K \geq c > 0$. Therefore, S is compact from the classical Bonnet-Myers theorem. Consequently, [2, Prop. 3.2], F must be also compact. Finally, S is a spacelike slice as a consequence of Theorem 3.1 according the assumption made on the warping function. \square

Corollary 3.6 *Let M be a GRW spacetime whose warping function satisfies $(\log f)'' \leq 0$. Let S a compact spacelike surface in M and put $\alpha = \min_S \frac{f'(t)^2}{f(t)^2}$. If the mean curvature function H of S satisfies*

$$H^2 \leq \alpha,$$

then S must be a spacelike slice. Moreover, there is no compact spacelike surface in M whose mean curvature function satisfies $H^2 < \alpha$.

Theorem 3.7 *Let M be a GRW spacetime, whose warping function satisfies $(\log f)'' \leq 0$ and whose fiber F has Gauss curvature $K^F \geq 0$. Then, there is no complete spacelike surface S in M such that for some positive constant ϵ , the inequality*

$$\frac{f'(t)^2}{f(t)^2} - H^2 > \epsilon > 0,$$

holds on all S .

Proof. Otherwise suppose there exists such a surface S . From (16), the Gauss curvature of S satisfies $K \geq \epsilon > 0$. Therefore, the Bonnet-Myers theorem asserts that S is compact. Consequently, [2, Prop. 3.2], F must be also compact. Moreover, as a consequence of Theorem 3.1, S is a spacelike slice with $H^2 = \frac{f'(t)^2}{f(t)^2}$, which is a contradiction. \square

Remark 3.8 As shown in [6, Th. 6.20], the only complete spacelike surfaces of constant mean curvature in the 3-dimensional steady state spacetime that are bounded away from the infinity future and whose mean curvature satisfies $H \leq 1$ are the spacelike slices. As application of the previous result, we have that there is no complete spacelike surface in the 3-dimensional steady state spacetime, whose mean curvature function satisfies $H^2 < 1 - \epsilon$, for a constant ϵ such that $0 < \epsilon < 1$.

Remark 3.9 a) The assumption on K^F in Theorem 3.7 is crucial to get the compactness of S thanks to the classical Bonnet-Myers theorem. However, if the compactness of S is directly assumed then, with no assumption on K^F , the following nonexistence result holds: if M is a proper (resp. non-necessarily proper) GRW spacetime, whose warping function satisfies

$(\log f)'' \leq 0$ (resp. $(\log f)'' < 0$), then there is no compact spacelike surface S in M such that its mean curvature function satisfies $\frac{f'(t)^2}{f(t)^2} - H^2 > 0$ everywhere on S . In fact, from (11) we have that there exists $p \in S$ such that $H^2(p) \geq \frac{f'(t(p))^2}{f(t(p))^2}$ (alternatively, p may be taken as point where $\log f(t)$ attains its minimum value). **b)** Previous argument implies that when inequality (\tilde{I}) is assumed on a compact spacelike surface S in a GRW spacetime M whose warping function satisfies $(\log f)'' \leq 0$, there is always some point of S where the inequality for H is indeed an equality. In the case S is complete and noncompact this is not true, in general.

We end this section with the case that the GRW spacetime M has constant sectional curvature \bar{c} . Note that this holds if and only if its fiber F has constant Gauss curvature $K^F = c$ (i.e., M is a RW spacetime) and its warping function f satisfies,

$$\frac{f''}{f} = \frac{c + (f')^2}{f^2} = \bar{c}, \quad (20)$$

(see for instance [4, Cor. 9.107]). All the positive solutions of (20) were found in [3], as a particular case of a much general result. Note that we have,

$$(\log f)'' = \frac{c}{f^2}.$$

directly from (20). As shown in [3, Table, cases 5,6], if $\bar{c} \leq 0$ then $c < 0$. Therefore, as a consequence of the previous results, we obtain the following extension of [3, Cor. 5] for the case of spacelike surfaces.

Theorem 3.10 *The only compact spacelike surfaces S in a RW spacetime with non-positive constant sectional curvature and whose mean curvature function H satisfies the inequality (\tilde{I}) on all S , are the spacelike slices.*

4 Uniqueness results of Calabi-Bernstein type

Let (F, g) be a 2-dimensional compact Riemannian manifold and let $f : I \rightarrow \mathbb{R}$ be a positive smooth function. For each $u \in C^\infty(F)$ such that $u(F) \subset I$ we can consider its graph $\Sigma_u = \{(u(p), p) : p \in F\}$ in the 3-dimensional GRW spacetime M with base $(I, -dt^2)$, fiber (F, g) and warping function f . The graph of u inherits a metric, given by (3), which is Riemannian if and only if u satisfies $|Du| < f(u)$ everywhere on F . Note that $t(u(p), p) = u(p)$ for any $p \in F$, and so, the functions t on Σ_u and u can be naturally identified. As an application of the previous results, we have the following uniqueness results,

Theorem 4.1 *Let F be a 2-dimensional compact Riemannian manifold and let $f : I \rightarrow]0, \infty[$ be a smooth function such that $(\log f)'' \leq 0$. Then, the only entire solutions $u : F \rightarrow I$ of the inequality (I) are the constant functions. Moreover, there is no entire solution of the inequality assuming that (I.1) holds strictly.*

Theorem 4.2 *Let F be a 2-dimensional compact Riemannian manifold and let $f : I \rightarrow]0, \infty[$ be a smooth function with $(\log f)'' \leq 0$ and such that $\inf \frac{(f')^2}{f^2} = \alpha > 0$. Then, the only entire solutions $u : F \rightarrow I$ of the inequality*

$$H(u)^2 \leq \alpha,$$

$$|Du| < f(u),$$

are the constant functions. Moreover, there is no entire solution if the first inequality holds strictly.

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